Continuous Random Variables

&

Uniform Probability Distribution

What is a **Random Variable**?

It is a quantity whose values are real numbers and are determined by the number of desired outcomes of an experiment.

Is there any special **Random Variable**?

We can categorize random variables in two groups:

- **Discrete** random variable
- **Continuous** random variable

What are **Discrete Random Variables**?

It is a numerical value associated with the desired outcomes and has either a finite number of values or infinitely many values but countable such as whole numbers $0, 1, 2, 3, \cdots$.

What are **Continuous Random Variables**?

It has infinitely many numerical values associated with any interval on the number line system without any gaps or breaks.

What is a **Probability Distribution**?

It is a description and often given in the form of a graph, formula, or table that provides the probability for all possible random variables of the desired outcomes.

Are there any **Requirements**?

Let x be any random variable and $P(x)$ be the probability of the random variable x, then

- $\sum P(x) = 1$
- $0 \leq P(x) \leq 1$

What is **Continuous Probability Distribution**?

It is a probability distribution for a continuous random variable x with probability $P(x)$ such that

- $\sum P(x) = 1$,
- $0 \leq P(x) \leq 1$, and
- $P(x = c) = 0$.

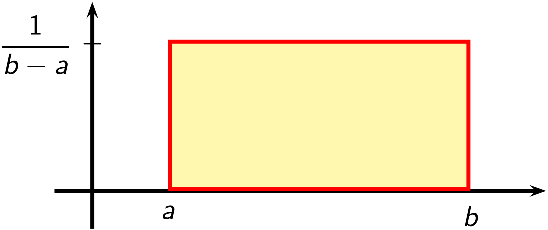
Here are few examples of **continuous probability distributions**:

- **Uniform** Probability Distribution
- **Standard Normal** Probability Distribution
- **Normal** Probability Distribution

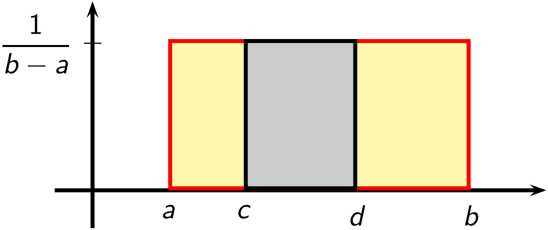
What is a **Uniform Probability Distribution**?

It is a probability distribution for a continuous random variable x that can assume all values on the interval $[a, b]$ such that

- All values are evenly spread over the interval $[a, b]$,
- The graph of the distribution has a rectangular shape,



- $P(x = c) = 0$, and
- $P(c < x < d)$ as shaded here



can be computed by simply finding the area of the shaded rectangular region.

$$P(c < x < d) \quad = \quad (d - c) \cdot \frac{1}{b-a}$$

Example:

The amount of time, in minutes, that a person must wait for a bus is from 0 and 12 minutes, inclusive, with a uniform probability distribution.

- Draw and label the uniform probability distribution,
- Find the probability that the wait time for a bus for a randomly selected person, is exactly 10 minutes.
- Find the probability that the wait time for a bus for a randomly selected person, is between 7.5 to 10 minutes.

Solution:

We have a uniform probability distribution with $a = 0$, $b = 12$, and a rectangular graph with the width of $\dfrac{1}{b-a} = \dfrac{1}{12}$.

Solution Continued:

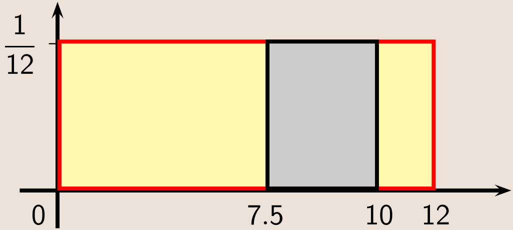
- Draw and label the uniform probability distribution,

- Find the probability that the wait time for a bus for a randomly selected person, is exactly 10 minutes.

$$P(x = 10) = 0$$

Solution Continued:

- Find the probability that the wait time for a bus for a randomly selected person, is between 7.5 to 10 minutes.



The gray region here represents the probability for wait time from 7.5 to 10 minutes.

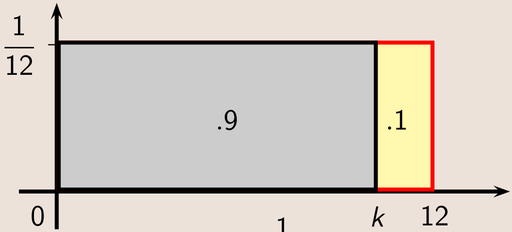
$$P(7.5 < x < 10) = (10 - 7.5) \cdot \frac{1}{12} = 2.5 \cdot \frac{1}{12} = \frac{5}{24}$$

Example:

Using the last example, find the time, rounded to one-decimal place, that separates the bottom 90% wait time from the rest.

Solution:

The gray area below is 0.9 since it represents $k = P_{90}$,



$$P(x < k) = (k - 0) \cdot \frac{1}{12}$$

$$0.9 = k \cdot \frac{1}{12} \Rightarrow k = 10.8$$

Example:

The amount of coffee dispensed by a certain machine into a cup is a continuous random variable x that assumes all values from $x = 11$ and $x = 25$ ounces with uniform probability distribution.

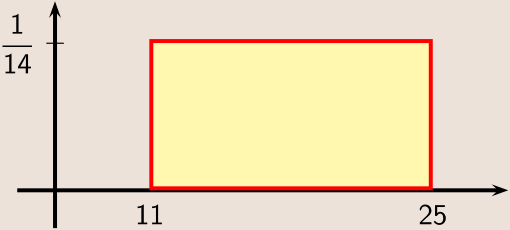
- Draw and label the uniform probability distribution,
- Find the probability that a cup filled by this machine will contain at least 14.5 ounces.
- Find the probability that a cup filled by this machine will contain at most 20 ounces.
- Find the probability that a cup filled by this machine will contain more than 16.5 ounces but less than 18 ounces.
- Find k such that $P(x > k) = 0.3$. Explain what this number represents.

Solution:

In this example, we have a uniform probability distribution with $a = 11$, $b = 25$.

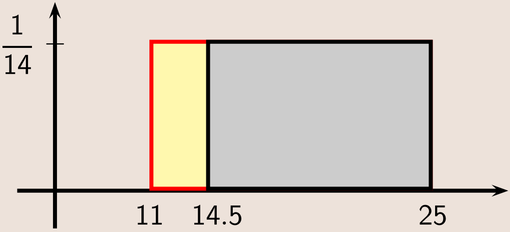
- Draw and label the uniform probability distribution

  The uniform probability distribution has a rectangular graph with its length going from $a = 11$ to $b = 25$, and the width of $\dfrac{1}{b-a} = \dfrac{1}{14}$.



Solution Continued:

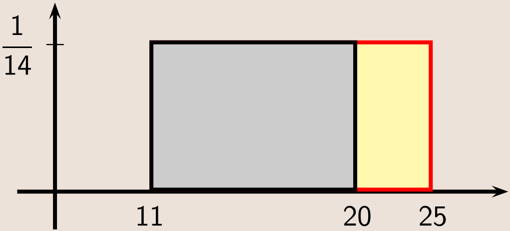
- Find the probability that a cup filled by this machine will contain at least 14.5 ounces.



$$
\begin{aligned}
P(x \geq 14.5) &= P(x > 14.5) \\
&= (25 - 14.5) \cdot \frac{1}{14} \\
&= 0.75
\end{aligned}
$$

Solution Continued:

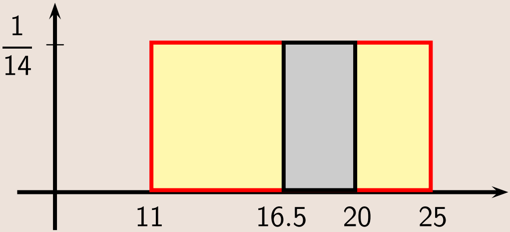
- Find the probability that a cup filled by this machine will contain at most 20 ounces.



$$\begin{aligned} P(x \leq 20) &= P(x < 20) \\ &= (20 - 11) \cdot \frac{1}{14} \\ &\approx 0.643 \end{aligned}$$

Solution Continued:

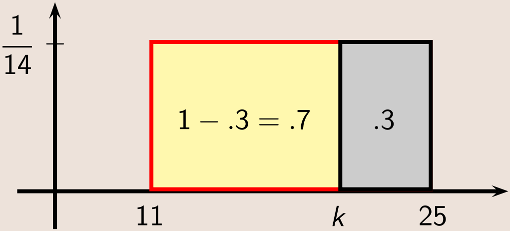
- Find the probability that a cup filled by this machine will contain more than 16.5 ounces but less than 18 ounces.



$$
\begin{aligned}
P(16.5 < x < 20) &= (20 - 16.5) \cdot \frac{1}{14} \\
&= (3.5) \cdot \frac{1}{14} \\
&= 0.25
\end{aligned}
$$

Solution Continued:

- Find k such that $P(x > k) = 0.3$. Explain what this number represents.



$$
\begin{aligned}
P(x > k) &= \text{Area of the gray region} \\
0.3 &= (25 - k) \cdot \frac{1}{14} \\
14 \cdot 0.3 &= 25 - k \\
4.2 &= 25 - k \Rightarrow k = 20.8
\end{aligned}
$$

Finding **Mean**, **Variance**, and **Standard Deviation** of a **Uniform Probability Distribution**:

Given a continuous random variable x that can assume all values on the interval $[a, b]$ with a uniform probability distribution, then

- Mean $\Rightarrow \mu = \dfrac{b+a}{2}$,
- Variance $\Rightarrow \sigma^2 = \dfrac{(b-a)^2}{12}$, and
- Standard Deviation $\Rightarrow \sigma = \sqrt{\sigma^2}$.

Example:

A continuous random variable x that assumes all values from $x = 5$ and $x = 30$ with uniform probability distribution.

- Draw and label the uniform probability distribution,
- Find the mean of the distribution.
- Find the variance of the distribution.
- Find the standard deviation of the distribution.

Solution:

We have a uniform probability distribution with $a = 5$, $b = 30$, and a rectangular graph with the width of $\dfrac{1}{b-a} = \dfrac{1}{25}$.

Solution Continued:

A continuous random variable x that assumes all values from $x = 5$ and $x = 30$ has a uniform probability distribution.

- Draw and label the uniform probability distribution,

- Find the mean of the distribution.

$$\begin{aligned} \mu &= \frac{b+a}{2} \\ &= \frac{30+5}{2} \Rightarrow \mu = 17.5 \end{aligned}$$

Solution Continued:

- Find the variance of the distribution.

$$\begin{aligned}\sigma^2 &= \frac{(b-a)^2}{12} \\ &= \frac{(30-5)^2}{12} \Rightarrow \sigma^2 = \frac{625}{12}\end{aligned}$$

- Find the standard deviation of the distribution.

$$\begin{aligned}\sigma &= \sqrt{\sigma^2} \\ &= \sqrt{\frac{625}{12}} \Rightarrow \sigma \approx 7.217\end{aligned}$$